(T1) Super Luminal Galaxies **(10 points)**

Read the statements given below and state if they are true or false:

(a) For some galaxies the apparent recession speed exceeds the speed of light.

(b) The velocity – Distance relation as given by Hubble cannot allow recession velocities to exceed the speed of light.

(c) Hubble-Lemaitre's law (formerly known as Hubble's Law) does not violate special relativity.

(d) If some galaxies would have an apparent recession speed exceeding the speed of light, then the photons from those galaxies can never reach us.

(e) As the expansion of Universe is accelerating, photons emitted right now from galaxies which have apparent recession speed equal to the speed of light will never reach us.

Solution:

T F T F F (2 points each)

(T2) Distance **(10 points)**

An observer measured trigonometric parallaxes of stars in a star cluster. Due to random errors, the measured parallax values are distributed symmetrically around the expected value with standard deviation equal to 0.05 mas (milliarcsec). Assume there are no systematic errors and assume all stars in the said cluster have the same luminosity. It is known that the distance of this cluster from us is R= 5 kpc.

He gave the data table to 4 of his students (A, B, C and D) and they estimated the distance to the cluster in the following ways:

A. Convert each parallax measurement into distance and then find the average distance (R_A)
B. Take the average of all parallaxes first and then convert the average parallax into distance. (R_B)
C. Convert each parallax measurement into distance and then take the median distance value. (R_C)
D. Find the median value of the parallaxes and then convert the median value into distance. (R_D)

State if the following statements are true or false. **In case a given mathematical relation is false, give the correct relation.**

(l) If the i^{th} star gave the smallest value of parallax and the j^{th} star gave the highest value of parallax, in all likelihood R_i-R > R-R_j

(m) R_A = R (i.e. there is a high chance that the distance estimated by A fairly matches the true distance)

(n) R_B = R (i.e. there is a high chance that the distance estimated by B fairly matches the true distance)

(o) R_C < R (i.e. there is a high chance that the distance estimated by C will be systematically lower than the true distance)

(p) R_D = R (i.e. there is a high chance that the distance estimated by D fairly matches the true distance)

Solution:

T F T F T (2 points each)

For parallaxes, we have a Gaussian with μ=0.2 mas and σ=0.05 mas i.e. 25% of the expectation value.

If we assume that the lowest and highest values of parallax are symmetric w.r.t. the expected value (reasonable assumption if cluster has large number of stars), then their distances would not be symmetric w.r.t. the true distance.

$R_A > R$, $R_C = R$ (for statement m and o, marks will be given only if this relation is stated)

(T3) Atmospheric Refraction **(10 points)**

Consider sunrise at Beijing (φ=40°) on the vernal equinox day.

(a) Let us say r_l, r_d, r_r and r_u are distances from the centre of the undistorted disk of the Sun to the edge of the disk towards the directions left, down, right and up respectively. What will be the hierarchical relation (<, =, >) between the four radii just after the sunrise?

(b) What is the correction in the time of rise of the top edge of the disk as compared to the case without atmosphere? You may assume that typically atmospheric refraction near the horizon is 35'. Please only consider the apparent diurnal motion.

Solution:

(a) $r_d < r_u < r_l = r_r$ (4 points)

(b) When the upper edge of the sun is under the horizon with 35′, it will appear on the horizon earlier due to the influence of atmospheric refraction. With the apparent diurnal motion, i.e.

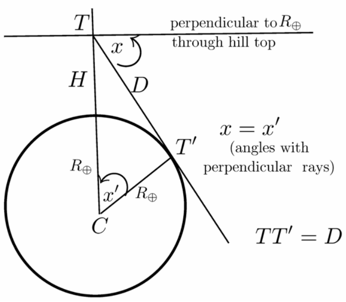
$$\omega = \frac{360^o}{23^h56^m4^s} = \frac{21600'}{1436.0682^m} = 15.0411\,'/m,$$ (2 points)

Ideally one should find time taken by mean sun to travel from 50' below horizon to 15' below true horizon.

$$\Delta t = \frac{(50'-15')}{\omega\cos\varphi} = \frac{35'}{15.0411'\times\cos 40°} = 3.04 \text{ mins}$$ (4 points)

(T4) Height of a Hill **(10 points)**

Two friends wanted to measure the height of the hill next to their village (latitude φ=40°). One of the friends climbed to the top of the hill and she agreed to send a light signal to her friend in the village as soon as she sees the sunset. On March 21, when they did this experiment, the friend in village received the light signal 4.1 minutes after the sunset from the village. Estimate the height of the hill and horizon distance for the person at the hill top. Ignore atmospheric refraction.



Solution:

First we should realise that the Sun doesn't set vertically. It sets at an angle of $\theta = 90 - \varphi = 50°$. Thus, the horizon depression *x* will be

$x = 15.0411' \times 4.1 \times sin\theta = 1.03^o sin50^o = 0.787^o$. (4 points)

Let the height of the mountain be H. Let *C* be the centre of the Earth, point *T* - hill top and point *T'* is the point on the horizon as seen from T. Thus,

$$cosx = \left[\frac{R_\oplus}{(R_\oplus + H)}\right]$$

$H = \frac{R_\oplus(1-cosx)}{cosx} = 602$ metres (3 points)

The distance of horizon (from base of the mountain, along the surface of the earth) is found from the rectangular triangle *TCT*' as

$D = (R_\oplus + H)sinx \approx R_\oplus sinx = 6.38 \times 10^3 sin0.787^o = 87.6$ km

Alternatively, $D = R_\oplus x = 6.38 \times 10^3 \times 0.787^o \times \pi \div 180^o = 87.6$ km (3 points)

(T5) Sidereal Time **(10 points)**

It is very interesting to observe that on one particular calendar day each year, the mean sidereal time will twice be 00:00:00.

(a) What will be the approximate R.A. of the Sun when this event happens?

(b) Estimate the exact date in 2018 for this event.

You may assume that at the Royal Greenwich Observatory, the mean sidereal time ($GMST_0$) was 6.706h at 0h, 1st January, 2018 (JD2458119.5).

Solution:

(1) This can happen if the sidereal time is 00:00:00 just after the midnight and just before the midnight on the same day. It means that the sun should be near the Autumn (September) Equinox. Approximate R.A. of the Sun is 18^h. (3 points)

(2) At 0^h, 1st January, 2018 (JD2458119.5), the mean sidereal time $GMST_0$ was 6.706^h, i.e. $GMST_0$=100.59°. Since one mean solar day is longer than one mean sidereal day with 0.9856º (about 4 minutes), the GMST will increase $\Delta = 0.9856°$at next midnight at 2st January, 2018. Therefore, we can first calculate how many days it will take for GMST to reach 360°, i.e.

$T = \frac{360° - GMST_0}{\Delta} = \frac{360° - 100.59°}{0.9856°} = 263.20$ (3 points)

i.e. after completion of 263 days, the GMST at midnight will be 359.80º.

Thus, GMST will be 00:00:00 on 264th day at 00:00:48 and then again at 23:56:52 on the same day.

(2 points)

The 264th day in the calendar of 2018 is 21st September 2018. (2 points)

(T6) Observe the Sun with FAST **(25 points)**

The Five-hundred-meter Aperture Spherical radio Telescope (FAST) is a single-dish radio telescope located in Guizhou Province, China. The physical diameter of the dish is 500 m, but during observations, the effective diameter of the collecting area is 300 m.

Consider observations of the thermal radio emission from the photosphere of the Sun at 3.0 GHz with this telescope and a receiver with bandwidth 0.3 GHz.

(a) Calculate the total energy ($E_\odot$) that the receiver will collect during 1 hour of observation.
(b) Estimate the energy needed to turn over one page of your answer sheet (E'). Hint: the typical surface density of paper is 80 gm^{-2}.
(c) Which one is larger?

Solution:

Rayleigh-Jeans law to calculate the thermal emission from the sun at 3 GHz states,

$$B_\nu = \frac{2k_B T}{c^2}\nu^2$$

which is the power emitted per unit emitting area, per steradian, per unit frequency. Therefore, the solar luminosity at 3 GHz should be:

$L_\nu = B_\nu \cdot 4\pi R_\odot^2$ (3 points)

At the distance of the earth (1 AU), the monochromatic flux from the sun at 3 GHz should be:

$$f_\nu = \frac{L_\nu}{4\pi D^2}$$

Hence the energy flux that FAST will receive is:

$$F_\nu = f_\nu \cdot \Delta\nu \cdot \pi^2 \frac{d_c^2}{4}$$ (8 points)

And the total energy that the receiver will collect during 1 hour of observation is:

$$E_\odot = F_\nu \Delta t = \frac{2k_B T}{c^2}\nu^2 \frac{R_\odot^2}{D^2} \cdot \pi^2 \frac{d_c^2}{4} \Delta\nu \Delta t = 8.5 \times 10^{-5} J$$ (8 points)

Then we can calculate the work we need to turn over one piece of the answer sheet (A4 paper). The mass of an A4 paper (297mm×210mm) is:

$$m = \rho \cdot L_1 L_2$$ (2 point)

Therefore, the energy of turning it over should be around:

$$E' = mg \cdot \frac{L_2}{2} \approx 5 \times 10^{-3} J$$ (3 points)

As a consequence, $E' > E_\odot$. (1 point)

(T7) Sunspot **(25 points)**

Magnetic fields are important in the physics of stars and sunspots. As an approximation, we can model the photosphere of the Sun consisting of a plasma, which can be simply treated as a single component ideal gas, and a magnetic field (**B**), which has an associated magnetic pressure $p_B = \frac{B^2}{2\mu_0}$. It behaves like any other physical pressure except that it is carried by the magnetic field rather than by the kinetic energy of particles.

Assume that the number density of particles in the photosphere is constant everywhere, but the magnetic field inside the sunspot (B_{in}=0.1T) is much stronger than outside ($B_{out}=5\times10^{-3}$T). From the blackbody spectrum, the temperature inside the sunspot is T_{in}~4000K, while the temperature outside is T_{out}~6000K (which is why the sunspot looks darker). For the sunspot to be stable, the inside must be in equilibrium with the outside.

(a) Estimate the number density of plasma particles in the solar photosphere.

(b) Compare your answer with an estimate of the number density of particles in the atmosphere at the surface of the Earth.

Solution:

(a) The pressure is from all directions. The plasma inside the sunspot must have the same total pressure as the plasma outside to maintain equilibrium on the border.

The kinetic pressure inside the sunspot is $p_i = n_i k_B T_i$, where n_i is the number density inside the sunspot. And also, $p_e = n_e k_B T_e$. (3 points)

($p_e V = N_e k_B T_e$ is also correct. But it's better to write the pressure in terms of number density)

From the assumption, $n_i = n_e = n$. The equilibrium requires: (3 points)

$$n k_B T_i + \frac{B_i^2}{2\mu_0} = n k_B T_e + \frac{B_e^2}{2\mu_0}$$ (8 points)

Then

$$n = \frac{1}{2\mu_0 k_B}\frac{(B_i^2 - B_e^2)}{(T_e - T_i)} = 1.43 \times 10^{23} m^{-3}$$ (6 points)

(b) On the Earth, $p_E = n_E k_B T_E$ where $P_E \approx 10^5$ Pa and $T_E \approx 300$ K.

Thus, $n_E = \frac{p_E}{k_B T_E} = 2.4 \times 10^{25} m^{-3}$ (4 points)

This means that the number density of the atmospheric particles at the surface of the earth is at least 100 times larger that the number density of particles in the solar photosphere (of course the mass density is even higher). (1 point)

(T8) A Possible Dark Matter Deficient Galaxy **(25 points)**

Earlier this year, a team of astronomers reported their discovery of a galaxy with much less dark matter than the galaxy evolution model predicted (van Dokkum et al. 2018, Nature). This galaxy, named NGC 1052-DF2, is located close to the elliptical galaxy NGC 1052 (D=20Mpc from the Sun) in the sky. The shape of NGC 1052-DF2 resembles an ellipse with semi major axis (a) of 22.6" and $\frac{b}{a} = 0.85$. Half of the total light from the galaxy comes from within this ellipse and the mean surface brightness within the ellipse is about 24.7 mag arcsec^{-2}.

(a) Calculate the total apparent magnitude of this galaxy.

(b) The team suggested the galaxy is a companion of NGC 1052. Determine the total mass of stars in NGC 1052-DF2, assuming it has a mass to light ratio $\left(\frac{M/M_\odot}{L/L_\odot}\right)$ of 2.0.

(c) The team identified 10 globular clusters in NGC 1052-DF2 with a mean galactocentric distance of 78.4". They also measured the velocity dispersion of these clusters to be not more than 8.4 km/s. Estimate the dynamical mass of this galaxy. For simplicity, assume the mass distribution in the galaxies is uniform and is spherically symmetric.

(d) This discovery was challenged by other groups (Kroupa et al., Nature, 2018, Truijlo et al., MNRAS, 2018), who claimed that NGC 1052-DF2 is not a satellite of NGC 1052, and it is located at a much smaller distance to us. Show why a smaller distance would weaken the assertion of the dark matter deficiency in NGC 1052-DF2.

Solution:

(1) Total area with in the ellipse (actually, half-light ellipse) of NGC 1052-DF2 is:

$A_{DF2} = \pi ab = 0.85\pi \times 22.6''^2 = 1363.9 arcsec^2$ (2 points)

Magnitude of the part of the galaxy within the ellipse:

$m_{ell} = SB - 2.5 \times log(A_{DF2}) = 24.7 - 2.5 \times log(1363.9) = 16.9$ (2 points)

Total magnitude of the whole galaxy:

$m_{gal} = m_{ell} - 2.5 \times log(2) = 16.1$ (2 points)

(2) Absolute mag of the galaxy:

$M_{0,gal} = m_{gal} + 5 - 5 \times log(D) = -15.4$ (2 points)

Convert to solar luminosity:

$\frac{L_{gal}}{L_\odot} = 10^{-0.4(M_{0,gal} - M_{0,\odot})} = 1.2 \times 10^8$ (2 points)

Thus, the total stellar mass should be:

$\frac{M_{gal}}{M_\odot} = \frac{L_{gal}}{L_\odot} \times \frac{M}{L} = 2.4 \times 10^8$ (2 points)

(3) Mean galactocentric distance of globular clusters:

$r_{gc} = \frac{\theta D}{206265} = 7.6$ kpc (2 points)

Using viral theorem:

$< 2K > + < U >= 0,$

$< K >= \frac{1}{2} M < v^2 >= \frac{1}{2} M\sigma^2,$

$\langle U \rangle = \frac{3}{5}\frac{GM^2}{r}$ (4 points)

Thus the maximal dynamical mass of this system should be:

$M_{dyn} = \frac{5 r_{gc}\sigma^2}{3G} = 2.1 \times 10^8 M_\odot$, even less than the total stellar mass within this radius. (3 points)

(4) From Question 2 and 3, we know:

$M_{star} \propto L_{gal} \propto F_{gal} \times D^2$

However, $M_{dym} \propto D$ (2 points)

Thus

$M_{dym}/M_{star} \propto 1/D$ (2 points)

If the distance measured is smaller by a certain factor, the stellar mass to dynamical mass ratio would increase by the same factor, thus the dark matter in NGC 1052-DF2 would not be as deficient as Dragonfly team claimed.

(T9) Radio Galaxy **(25 points)**

An observer wants to use the Five-hundred-meter Aperture Spherical radio Telescope (FAST) in China to observe a radio galaxy at redshift of $z = 0.06$. We assume that the radio source is compact compared to the beam size of the telescope at the observing frequencies, i.e., the source is point-like as seen through the telescope. To detect a point source with FAST, it must be sufficiently strong (bright) relative to the noise level (for single polarization observations), σ, which depends on the bandwidth, $\Delta\nu$, and the integration time (the radio astronomy equivalent of exposure time), t_i, as follows:

$$\sigma = \frac{2k_B T_{sys}}{A_e\sqrt{t_i \Delta\nu}}$$

where T_{sys} is the system temperature (about 150 K in the frequency range of 0.28 GHz – 0.56 GHz and 25 K in the frequency range of 1.05 GHz – 1.45 GHz), and $A_e = 4.6\text{x}10^4\ \text{m}^2$ is the effective area of the telescope taking into account the total efficiency of the instrument.

This radio galaxy has an observed continuum flux density of $f_\nu = 2.5 \times 10^{-3} Jy$ at an observing frequency of 0.4 GHz. The bandwidth $\Delta\nu$ for the continuum observation centered at 0.4 GHz is 2.8×10^8 Hz.

(a) In order to detect the continuum flux density at 0.4 GHz with a signal-to-noise ratio of 30 (a so-called 30σ detection), what is the required integration time, t_i?

(b) We want to search for the neutral Hydrogen (HI) in the galaxy using 21cm absorption line. The HI 21cm line, with rest frame frequency of 1.4204 GHz. Calculate the observed frequency (υ_{obs}) of the HI line for this galaxy.

(c) The radio continuum emission from this galaxy can be described by a power law $f_\nu \sim \nu^\alpha$, with a spectral index of $\alpha = -0.2$. Calculate the continuum flux density at ν_{obs} for this galaxy.

(d) The line width of the HI 21cm absorption line is 90 km/s. Calculate the line width in Hz at the observing frequency of ν_{obs}. According to Figure 1, the HI 21cm line absorbs 4% of the continuum flux density (on average) over the line width of 90 kms^{-1}. In order to detect the absorption line at ≥3σ in three consecutive 30 kms^{-1} channels, what is the required integration time?

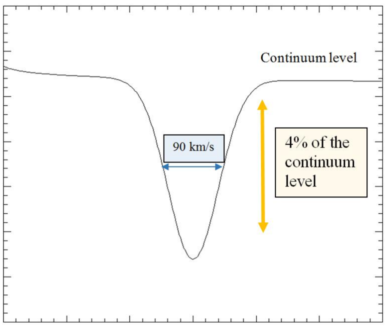


Figure 1: Spectrum of the HI 21cm absorption relative to the continuum emission in the radio galaxy

Solution:

(a) for signal to noise ratio of 30 of flux density of 2.5×10^{-3} Jy,

$\sigma = 2.5\times10^{-3}/30$ Jy $\approx 8.33\times10^{-5}$ Jy $= 8.33\times10^{-31}$ Wm^{-2}Hz^{-1} (4 points)

$T_{sys} = 150$ K , $\Delta\nu = 280$ MHz,

Using the equation above, $t_{int} \sim 42$ s (more exactly 41.8 s) (3 points)

(b) $\nu_{obs} = 1.4204$ GHz / $(1+z) = 1.4204/1.06 = 1.34$ GHz (3 points)

(c) $f_{vobs=1.34\text{GHz}} = f_{0.4\text{GHz}}\left(\frac{1.34}{0.4}\right)^{-0.2} = 2.5\ \times 10^{-3}\text{Jy} \times \left(\frac{1.34}{0.4}\right)^{-0.2} = 1.96\ \times 10^{-3}\text{Jy}$ (3 points)

(d) Line width : 90 kms^{-1} / $c \times \nu_{obs} = 90 / (2.9979\times10^{5}) \times 1.34$ GHz $= 0.402$ MHz (2 points)

4% of 1.96×10^{-3} Jy continuum $= 7.84\times10^{-5}$ Jy (1 point)

$\geq 3\sigma$ means $\sigma = 7.84\times10^{-5}$ Jy / 3 $= 2.61\times10^{-5}$ Jy $= 2.61\times10^{-31}$ Wm^{-2}Hz^{-1} (values around 2.6×10^{-5} Jy are ok)

in three consecutive 30 kms^{-1} channels: $\Delta\nu = 0.402$ MHz/3 $= 0.134$ MHz (6 points)

$T_{sys} = 25$ K

Using equation (1), $t_{int} = 24700\text{s} \approx 6.9$ hours (3 points)

(T10) Vega and Altair **(75 points)**

As per a very famous Chinese folklore about love, Vega and Altair are two lovers. It is said that they can meet each other once every year on a bridge made up of birds over the Milky Way. The parameters of two stars are given in the table below. For the purpose of this question, assume that the coordinate frame is fixed (i.e. not affected by precession or motion of the Sun).

Star	Right Ascension (J2000.0)	Declination (J2000.0)	Parallax (mas)	Proper Motion		Radial Velocity (km/s)
				$\mu_\alpha \cos\delta$ (mas/year)	μ_δ (mas/year)	
Vega	$18^h36^m56.49^s$	+38° 47' 07.7"	130.23	+200.94	+286.23	-13.9
Altair	$19^h50^m47.70^s$	+8° 52' 13.3"	194.95	+536.23	+385.29	-26.1

Based on this data, answer the following questions：
(a) (9 points) What is the angular separation of the two stars?
(b) (6 points) Calculate the distance (in parsecs) between Vega and Altair.
(c) (3 points) Calculate position angles of the proper motion vectors of each of these two stars.

For parts d-g, assume that the angular velocity of the stars on the celestial sphere remains constant. This is not a physical situation but this is an assumption to simplify the problem.
(d) (2 points) How many common points on the celestial sphere are there which can be reached by both these stars?
(e) (20 points) Find the coordinates of the closest such point.
(Note: Drawing the situation on a celestial sphere will help you in visualising the situation)
(f) (8 points) Find when (which year) each of these stars were / will be at that point.
(g) (5 points) When Altair was / will be at that point, what would be its angular separation from Vega?
(h) (22 points) Find coordinates of any point (if it exists) in 3-D space which was /will be visited by both these stars. Do not ignore radial velocities for this part of the question.

Solution:

(a) Assuming the positions of the two starts are: Vega: (α_1,δ_1); Altair : (α_2,δ_2)

$(\alpha_1,\delta_1) = (279.23538^o, 38.78547^o)$ and $(\alpha_2,\delta_2) = (297.69875^o, 8.87036^o)$ (2 points)

Cosine formula of the spherical geometry is given by:

$$cos[a] = cos[b]cos[c] + sin[b]sin[c]cos[A] \quad \text{(0.5 point)}$$

In our triangle, b = $(90 - \delta_1)$, c = $(90 - \delta_2)$ and A = $(\alpha_2 - \alpha_1)$. Then the angular distance *r*d between Vega and Altair is:

$$cos[\beta] = cos[90^o - \delta_1]cos[90^o - \delta_2] + sin[90^o - \delta_1]sin[90^o - \delta_2]cos[\alpha_2 - \alpha_1]; \quad \text{(3 points)}$$

$$cos[\beta] = cos[51.21453^o]cos[81.12964^o] + sin[51.21453^o]sin[81.12964^o]cos[18.46337^o];$$

(1.5 points)

$\beta = 34.19582^o$ (2 points)

(b) The distance of Vega is: $r_1 = 1/(130.23 \times 10^{-3}) = 7.6787pc$

The distance of Altair is: $r_2 = 1/(194.95 \times 10^{-3}) = 5.1295pc$ (2 points)

Using the Law of Cosines： $d^2 = {r_1}^2 + {r_2}^2 - 2r_1r_2cos[\beta]$ (2 point)

we can obtain the distance: $d = 4.4855pc$ (2 points)

(c) Given the proper motions and radial velocities of the two stars, the directions of their movements on the celestial sphere can be estimated.

For Vega:

$$\mu_{\alpha 1}cos\delta_1 = 200.94; \mu_{\delta 1} = 286.23$$

$$\theta_1 = arctan\left[\frac{\mu_{\alpha 1}cos\delta_1}{\mu_{\delta 1}}\right] \times \frac{180^o}{\pi} = 35.0697^o \quad \text{(2 points)}$$

For Altair:

$$\mu_{\alpha 2}cos\delta_2 = 536.23; \mu_{\delta 2} = 385.29$$

$$\theta_2 = arctan\left[\frac{\mu_{\alpha 2}cos\delta_2}{\mu_{\delta 2}}\right] \times \frac{180^o}{\pi} = 54.302^o \quad \text{(1 points)}$$

(d) As the position angles of the proper motion of two stars is different, the stars' paths will intersect. As the paths are circular on celestial sphere, they will intersect in exactly **<u>two</u>** points.

(2 points)

(e) Let the closer point of intersection be I with the coordinates (α_3,δ_3).

In triangle PVA, $PAV = arcsin\left[\frac{sinVPAsinPV}{sinVA}\right] = arcsin\left[\frac{sin(\alpha_1-\alpha_2)sin(90^o-\delta_1)}{sin\beta}\right] = 26.055^o$

$$PVA = arcsin\left[\frac{sinVPAsinPA}{sinVA}\right] = arcsin\left[\frac{sin(\alpha_1-\alpha_2)sin(90^o-\delta_2)}{sin\beta}\right] = 146.196^o \quad \text{(5 points)}$$

Note that angle PVA is more than 90°, which will be evident from the diagram. (1 point)

The motion of the stars on the celestial sphere will always be along some great circle. So VAI is a spherical triangle. Using four parts formula for VAI

$$cotVI = \frac{cosVAcosIVA + sinIVAcotVAI}{sinVA}$$

$$cotVI = \frac{cosVAcosIVA + sinIVAcotVAI}{sinVA}$$

$$VI = 90^o - arctan\left[\frac{cos34.19582^o cos(68.900^o) + sin(68.900^o)cot(99.643^o)}{sin34.19582^o}\right] = 76.085^o$$

(5 points)

by sine rule, $AI = arcsin\left[\frac{sinVIsinIVA}{sinVAI}\right] = arcsin\left[\frac{sin76.085^o sin68.900^o}{sin99.643^o}\right] = 66.715^o$

(2 points counted in next part)

Now we use triangle PVI

by cosine rule, $cos[PI] = cos[PV]cos[VI] + sin[PV]sin[VI]cos[PVI]$;

$cos[90^o - \delta_3] = cos[90^o - \delta_1]cos[VI] + sin[90^o - \delta_1]sin[VI]cos[180^o - \theta_1]$;

$$sin[\delta_3] = cos[51.215^o]cos[76.085^o] + sin[51.215^o]sin[76.085^o]cos[144.930^o];$$

Thus, $\delta_3 = -27.945^o$ (4 points)

Applying sine rule, $VPI = arcsin\left[\frac{sinPVIsinVI}{sinPI}\right] = arcsin\left[\frac{sin144.930^o sin76.085^o}{sin117.945^o}\right] = 39.148^o$

Hence, $\alpha_3 = \alpha_1 - VPI = 279.235^o - 39.148^o = 240.087^o \approx 16^h 0^m 21^s$ (5 points)

Out of 5 points, 1 point is reserved for realising that the intersection point is in past path.

(f) The total velocity along the great circle for Vega is

$\mu_1 = \sqrt{(\mu_{\alpha 1} cos\delta_1)^2 + \mu_{\delta 1}{}^2} = \sqrt{200.94^2 + 286.23^2} = 349.72$ mas/year (2 points)

Number of years $= \frac{VI}{\mu_1} = \frac{76.085 \times 3600}{0.34972} \approx 783200$years

It will happen in the year 781200 BCE (2 points)

Similarly, $\mu_2 = 660.30$and number of years =363700 years

It will happen in the year 3617 BCE (2 points)

(g) After 363700 years, Vega would have traversed 35.335° along its path. (3 points)

Thus, its separation from Altair will be (76.085° – 35.335°) = 40.750° (2 points)

(h) In earth centric Cartesian frame, coordinates of Vega (in units of parsec) will be:

$x_1 = r_1 cos\delta_1 cos(360^o - \alpha_1) = 0.96062$pc

$y_1 = r_1 cos\delta_1 sin(360^o - \alpha_1) = 5.90793$pc

$z_1 = r_1 sin\delta_1 = 4.80998$pc

Similarly, coordinates of Altair will be, x_2= 2.35579, y_2= 4.48736 pc and z_2= 0.79097 pc.
(6 points)

Here, we are assuming the North as positive x-direction and pole as positive z-direction.

The direction vectors for velocities in spherical coordinates are

$v_1 = (-13.9, 7.33, 10.45)$and $v_2 = (-26.9, 19.57, 14.06)$ (4 points)

The same in Cartesian coordinates will be

$$v_{x1} = v_{r1} cos\delta_1 cos\alpha_1 - v_{\alpha 1} sin\alpha_1 - v_{\delta 1} sin\delta_1 cos\alpha_1 = 4.45$$

$$v_{y1} = -v_{r1} cos\delta_1 sin\alpha_1 - v_{\alpha 1} cos\alpha_1 + v_{\delta 1} sin\delta_1 sin\alpha_1 = -18.33$$

$$v_{z1} = v_{r1} sin\delta_1 + v_{\delta 1} cos\delta_1 = -0.56$$

Similarly, Cartesian velocity vector of Altair will be, v_2= (4.33, -33.85, 9.87). (8 points)

Lastly, we evaluate the value of determinant,

$$\begin{vmatrix} (x_1-x_2) & (y_1-y_2) & (z_1-z_2) \\ v_{x1} & v_{y1} & v_{z1} \\ v_{x1} & v_{y1} & v_{z1} \end{vmatrix} = \begin{vmatrix} -1.40 & 1.42 & 4.02 \\ 4.45 & -18.33 & -0.56 \\ 4.33 & -33.85 & 9.87 \end{vmatrix}$$

= 279.82 – 65.81 – 286.48 = -72.47

As the value of the determinant is non-zero, the direction vectors of these two stars do not cross each other. Hence no such point can exist. (4 points)

(T11) Thermal History of the Universe (75 points)

Based on Einstein's general relativity, Russian physicist Alexander Friedmann derived the Friedmann Equation by which the dynamics of a homogeneous and isotropic universe can be well described. The Friedmann Equation is usually written as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}.$$

We define the Hubble parameter as $= \frac{\dot{a}}{a}$, where a is the scale factor and $\dot{a}$ is the rate of change of scale factor with time. Thus, the Hubble parameter is a function of cosmic time. In the Friedmann Equation, ρ_m is the density of matter, including dark matter and baryons, ρ_r is the density of radiation, Λ is the cosmological constant, and k is the curvature of space. Subscript 0 indicates the value of a physical quantity at present day, e.g. H_0 is the present value Hubble parameter. Also, to avoid confusion with the reduced Hubble parameter, we use the reduced Planck Constant $\hbar = h/(2\pi)$ instead of the Planck constant h.

(a) (5 points) What are the dimensions of Hubble parameter? One can define a characteristic timescale for the expansion of the Universe (i.e. Hubble time t_H) using the Hubble parameter. Calculate the present-day Hubble time t_{H0}.

(b) (5 points) Let us define the critical density ρ_c as the matter density required to explain the expansion of a flat universe without any radiation or dark energy. Find an expression of the critical density, in terms *H* and *G*. Calculate the present critical density ρ_{c0}.

(c) (6 points) It is convenient to define all density parameters in a dimensionless manner like $\Omega_i = \frac{\rho_i}{\rho_c}$, i.e. the ratio of density to critical density. The Friedmann Equation can be rewritten using these dimensionless density parameters simply as, $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$.

Use this information to find expression for Ω_Λ and Ω_k, in terms *H, c, Λ, k* and *a*.

(d) (7 points) Another equation which is valid for matter, radiation and dark energy is often called the Fluid Equation: $\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$, where p is the pressure of some component, ρ is the density and $\dot{\rho}$ is the rate of change of density over time. Radiation contains photons and massless neutrinos, and they both travel at the speed of light. The pressure exerted by these particles is 1/3 of their energy density. Show that the density of radiation $\rho_r \propto (1+z)^4$, where z is cosmological redshift. You may note that if $\frac{\dot{\rho}}{\rho} = n\,\frac{\dot{a}}{a}$, then $\rho \propto a^n$

(e) (4 points) We know that the value of the cosmological constant Λ doesn't evolve. Its equation of state has a form $p = w\rho_\Lambda c^2$, where *w* is an integer. Find the value of *w*.

(f) (13 points) Planck time, defines a characteristic timescale before which our present physical laws are no longer valid, and where quantum gravity is needed. The expression for Planck time can be written in terms of $\hbar$, G and c and non-dimensional coefficient of this expression in SI units is of the order of unity. Using dimensional analysis, find expression for Planck time and estimate its value.

(g) (7 points) Planck length defines the length scale associated with Planck time is given by $l_P = ct_P$. The minimal mass of a black hole, also called Planck mass, is defined as the mass of a black hole whose Schwarzschild radius is two times the Planck length.

Derive the Planck mass M_P and calculate $M_P c^2$ in GeV. This mass is considered to be an upper threshold for elementary particles, beyond which they will collapse to a black hole.

(h) (4 points) At the very beginning (soon after the Planck time), all the particles were in thermal equilibrium in a primordial soup. As temperature decreased, different particles then decoupled from the primordial soup one by one and could travel freely in the Universe. Photons decoupled at ~300000 years after the Big Bang. These photons emitted at that time are what constitutes the cosmic microwave background (CMB), which follows the Stefan-Boltzmann law for blackbody radiation.

$$\varepsilon_r = \frac{\pi^2}{15\hbar^3 c^3}(k_B T)^4,$$

Show that the temperature of the CMB follows $T/(1+z) = constant$.

(i) (16 points) With the expansion of the Universe, radiation density dropped more quickly than matter density, and at some epoch the matter density was equal to the radiation density. Radiation contains both photons and neutrinos. Apart from photons, neutrinos additionally contribute to the radiation energy density by 68% (i.e. $\Omega_{r0} = 1.68\Omega_{\gamma 0}$, where γ indicates photons). Estimate the redshift of matter-radiation equality z_{eq} in terms of Ω_{m0} and reduced Hubble parameter $h = \frac{H_0}{100km\,s^{-1}Mpc^{-1}}$ You may use the current temperature of the CMB: $T_0 = 2.73K$.

(j) (8 points) The neutrinos decoupled from the primordial soup when the temperature of the universe was around 1 MeV. At this time, the radiation density in the universe was much more than all other components. Estimate the time ($t = \frac{1}{2H}$) when neutrinos decoupled, and express it in seconds since the big bang.

Solution:

(a) One can argue this in different ways. One can note that the Hubble parameter is often called Hubble constant, but actually it's not a constant. The dimensions of Hubble parameter is the inverse of time $[T^{-1}]$. Alternatively, one can simply look at unit of H_0 in the table of constants and conclude the same. *(1 point)*

It's natural to define a timescale as the reciprocal of Hubble parameter: $t_H = \frac{1}{H}$, and this is the Hubble time which is a characteristic timescale of Universe expansion.
(2 points)

Present-day Hubble time t_{H0} =14.46 Gyr. (2 points)

(b) From Friedmann Equation, the critical density is defined in the way of $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_c$, thus $\rho_c = \frac{3H^2}{8\pi G}$.

(2 points)

Substitute $H_0 = 2.19 \times 10^{-18}\ s^{-1}$ into above, we have $\rho_{c0} = 8.59 \times 10^{-27}\ kg/m^3$ (3 points)

(c) $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$

$$\frac{\rho_m}{\rho_c} + \frac{\rho_r}{\rho_c} + \Omega_\Lambda + \Omega_k = 1$$

$$\frac{8\pi G}{3H^2}(\rho_m + \rho_r) + \Omega_\Lambda + \Omega_k = 1$$

$$\frac{8\pi G}{3}(\rho_m + \rho_r) + H^2\Omega_\Lambda + H^2\Omega_k = H^2$$ (2 points)

Comparing this with the Friedmann equation $H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$.

$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$ and $\Omega_k = -\frac{kc^2}{a^2H^2}$ (4 points)

(d) Pressure exerted by the radiation is $p = \frac{1}{3}\rho_r c^2$

Thus, the Fluid Equation becomes, $\dot{\rho_r} + 3\frac{\dot{a}}{a}\left(\rho_r + \frac{1}{3}\rho_r\right) = 0$ (2 points)

$\frac{\dot{\rho_r}}{\rho_r} = -4\frac{\dot{a}}{a}$ implying

Thus $\rho_r \propto a^{-4}$. (3 points)

And we know $a = \frac{1}{1+z}$, hence $\rho_r \propto (1+z)^4$. (2 points)

(e) In the fluid equation, $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + q\rho) = 0$.

But the cosmological constant does not evolve. So $\dot{\rho_\Lambda} = 0$

$\frac{3\dot{a}\rho_\Lambda}{a}(1+q) = 0$ which means q = -1. (4 points)

(f) From the units, one can figure out that $\hbar$ has the dimensions of $[ML^2T^{-1}]$.

G has the dimensions of $[M^{-1}L^3T^{-2}]$.

And speed of light c has dimensions of $[LT^{-1}]$. *(5 points)*

We assume that Planck time is given by $\hbar^x c^y G^z$. In order to make it a time dimensional quantity, the following equation must hold:$T = (ML^2T^{-2})^x(LT^{-1})^y(M^{-1}L^3T^{-2})^z. T = M^{x-z}L^{2x+y+3z}T^{-x-y-2z}$.

(3 points)

From $x - z = 0, 2x + y + 3z = 0, -x - y - 2z = 1$, we get $x = \frac{1}{2}, y = \frac{-5}{2}, z = \frac{1}{2}$. Thus, the Planck time is approximately $t_p \approx \sqrt{\hbar G/c^5}$. (3 points)

Substitute the numerical values into it and get:$t_P = 5.4 \times 10^{-44}s$. (2 points)

(g) Schwarzschild radius of a black hole equals to $2l_P$: $r_s = \frac{2GM_P}{c^2} = 2ct_P = 2c\sqrt{\hbar G/c^5}$, (3 points)

$M_P = \sqrt{\hbar c/G}$. (2 points)

$M_P c^2 = \sqrt{\hbar c^5/G} = 1.22 \times 10^{19} GeV$. (2 points)

(h) Given the energy density, and the result of problem **d**, we find that $\varepsilon_r \propto T^4, \varepsilon_r = \rho_r c^2, \rho_r \propto (1+z)^4$.

Thus $T \propto 1+z, \frac{T}{1+z} = const.$ (4 points)

(i) The matter-radiation equality means $\Omega_r(z_{eq}) = \Omega_m(z_{eq})$. We need to derive the behavior of this two density parameters. Obviously $\Omega_m(z) = \Omega_{m0}(1+z)^3$ and $\Omega_r(z) = \Omega_{r0}(1+z)^4$. (4 points)

Thus $1 + z_{eq} = \frac{\Omega_{m0}}{\Omega_{r0}}$. Ω_{r0} is the density parameter of CMB radiation and can be calculated. (1 point)

By the definition of density parameter of photon radiation (footnote gamma: γ, $\Omega_{\gamma 0} = \frac{\rho_{\gamma 0}}{\rho_{c0}} = \frac{\varepsilon_{\gamma 0}}{c^2}\frac{8\pi G}{3H^2} = \frac{\varepsilon_{\gamma 0}}{c^2}\frac{8\pi G}{3H_{100}^2 h^2}$, where $H_{100} = 100\, km/s/Mpc$. (4 points)

The energy density of blackbody photon radiation is $\varepsilon_\gamma = \frac{\pi^2}{15\hbar^3c^3}(k_BT)^4$, thus

$$\Omega_{\gamma 0}h^2 = \frac{\pi^2}{15\hbar^3c^5}k_B^4\frac{8\pi G}{3H_{100}^2}T_0^4 = 2.47 \times 10^{-5}.$$ (4 points)

But there is also an additional contribution from neutrinos of about 68%, so the total density parameter of radiation is$\Omega_{r0}h^2 = \Omega_{\gamma 0}h^2 \times 1.68 = 4.15 \times 10^{-5}$. (2 points)

Hence $1 + z_{eq} = \frac{\Omega_{m0}}{\Omega_{r0}} = 2.4 \times 10^4 \Omega_{m0} h^2$. (1 point)

(j) At that time, the redshift is

$$1 + z = \frac{1MeV}{k_B \times 2.73K} = 4.25 \times 10^9.$$

Thus the scale factor $a = \frac{1}{1+z} = 2.35 \times 10^{-10}$. (2 points)

Only considering radiation dominated Universe, Friedmann Equation can be written as

$\left(\frac{\acute{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_r = \frac{8\pi G \rho_{r_0}}{3a^4}$, (2 points)

$\frac{\acute{a}}{a} = \frac{1}{a^2} \sqrt{\frac{8\pi G \rho_{r0}}{3}}$

$a^2 H = \sqrt{\frac{8\pi G \Omega_{r0}}{3} \rho_{c0}} = \sqrt{H_0^2 \Omega_{r0}}$ (1 points)

Thus, we have, $t = \frac{1}{2H} = \frac{a^2}{2\sqrt{H_0^2 \Omega_{r0}.}} = 1.3s \sim 1s.$

Thus neutrino decoupled at about 1 s after big bang. (3 points)